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Lexicographic Orthogonality

D. J. FOULIS AND C. H. RANDALL

Department of Mathematics, University of Massachusetts, Amherst, Massachusetts 01002

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We give a new construction showing how new orthomodular lattices can be built out of old ones by choosing any join dense subset of the given orthomodular lattice and putting a lexicographic orthogonality relation on the free monoid generated by this subset. This construction has relevance to the logic of an empirical science and, in particular, to the logic of quantum mechanics.

1. INTRODUCTION

Let L denote any complete orthomodular lattice [2]. A subset X of L is said to be *join dense* in L if every element of L is the join of the elements in X that it dominates. Let X be a join dense subset of L such that $0 \notin X$ and define the relation $\perp$ on X by $x \perp y$ if and only if $x \leqslant y'$ in L. Evidently, the relation $\perp$ is symmetric and antireflexive on X. If M is a subset of X, we define $M^{\perp} = \{x \in X \mid x \perp m \text{ for all } m \in M\}$ and we define $M^{\perp\perp} = (M^{\perp})^{\perp}$. We call a subset M of X *closed* in case $M = M^{\perp\perp}$. If $a \in L$, define $X_a = \{x \in X \mid x \leqslant a\}$, noting that $X_{a'} = (X_a)^{\perp}$ and that X_a is closed. The set $\mathscr{C}(X)$ of all closed subsets of X, partially ordered by set-theoretic inclusion and with the mapping that assigns to each closed set M the closed set $M^{\perp}$ as orthocomplementation, forms a complete orthomodular lattice that is isomorphic to the original one L under the correspondence $a \leftrightarrow X_a$.

Suppose, now, that X is any non-empty set and that $\perp$ is a symmetric antireflexive binary relation on X. The pair $(X, \perp)$ will be called an *orthogonality space*. For a subset M of X, we can still define $M^{\perp}$ as above and we can still look at the set $\mathscr{C}(X, \perp) = \{M \subset X \mid M = M^{\perp\perp}\}$ of all closed subsets of X partially ordered by inclusion. It is easy to check that, with $M \mapsto M^{\perp}$ as the orthocomplementation, $\mathscr{C}(X, \perp)$ is a complete ortho-lattice [2]; but, in general, $\mathscr{C}(X, \perp)$ need not be orthomodular. We shall say that $(X, \perp)$ is a *complete orthomodular space* if and only if the complete ortho-lattice $\mathscr{C}(X, \perp)$ is orthomodular.

A study of the calculus of "events" associated with the execution of physical operations [4] has led to the following construction: Let $(X, \#)$ be any orthogonality space (henceforth called the *base space*). Let Γ denote the free monoid (semigroup with unit 1) over the set X. For $a, b \in \Gamma$, say that a is *orthogonal* to b and write $a \perp b$ if and only if there exist elements $e, f, g \in \Gamma$ and elements $x, y \in X$ such that $x \# y$, $a = exf$ and $b = eyg$. Evidently $(\Gamma, \perp)$ is again an orthogonality space. For the purposes of empirical logic it is important to know that, if $(X, \#)$ is a complete orthomodular space, then so is $(\Gamma, \perp)$. The purpose of this paper is to prove this result.

2. Complete Orthomodular Spaces

We begin by giving some simple examples of complete orthomodular spaces. If X is any non-empty set, then $(X, \neq)$ is a complete orthomodular space called the *classic orthogonality space* over X. Note that $\mathscr{C}(X, \neq)$ is the complete atomic Boolean algebra of all subsets of X. Let H be any pre-Hilbert space and put $X = H \setminus \{0\}$. For vectors $x, y \in X$, define $x \perp y$ as usual. By a theorem of Amemiya and Araki, $(X, \perp)$ is a complete orthomodular space if and only if H is a Hilbert space [1]. Let $(Z_\alpha, \perp_\alpha)$ be a family of complete orthomodular spaces indexed by $\alpha \in A$. Suppose that $\alpha, \beta \in A$ with $\alpha \neq \beta$ implies $Z_\alpha \cap Z_\beta = \varnothing$. Put $Z = \bigcup_\alpha Z_\alpha$ and define $\perp$ on Z as follows: For $x, y \in Z$, $x \perp y$ if and only if there exists $\alpha \in A$ such that $x, y \in Z_\alpha$ and $x \perp_\alpha y$. Then $(Z, \perp)$ is a complete orthomodular space (called the *disjoint sum* of the spaces Z_α, $\alpha \in A$).

If $(Z, \perp)$ is an orthogonality space and if $D \subset Z$, then we call D an *orthogonal* subset of Z in case, for $x, y \in D$ with $x \neq y$, we always have $x \perp y$. If A is a closed subset of Z and if D is a maximal orthogonal subset of A, then it is natural to ask whether or not $D^{\perp\perp} = A$. Unless $(Z, \perp)$ is a complete orthomodular space, we will not generally have $D^{\perp\perp} = A$. In fact, we have the following theorem:

Theorem 1. *Let $(Z, \perp)$ be any orthogonality space. Then the following three conditions are mutually equivalent:*

(i) *$(Z, \perp)$ is a complete orthomodular space.*

(ii) *If D is an orthoogn al subset of Z, if $z \in Z$, if $z \notin D^\perp$ and if $z \notin D^{\perp\perp}$, then $D^\perp \cap (z^\perp \cap D^\perp)^\perp \neq \varnothing$.*

(iii) *If A is a closed subset of Z and if D is a maximal orthogonal subset of A, then $D^{\perp\perp} = A$.*

Proof. Suppose that (i) holds. Then $\mathscr{C}(Z)$ is a complete orthomodular lattice and the infimum of a family of elements of $\mathscr{C}(Z)$ is just their set-theoretic intersection. Also, if B is any subset of Z, then $B^{\perp} \in \mathscr{C}(Z)$. Hence, if we have $D^{\perp} \cap (z^{\perp} \cap D^{\perp})^{\perp} = \varnothing$, we have $D^{\perp} \wedge (z^{\perp\perp} \vee D^{\perp\perp}) = \varnothing$ $=$ the order zero in $\mathscr{C}(Z)$. Elementary orthomodular lattice theory allows us to conclude that $z^{\perp\perp} \subset D^{\perp\perp}$, that is, $z \in D^{\perp\perp}$. Hence (i) $\Rightarrow$ (ii).

Suppose that (ii) holds, that A is a closed subset of Z, and that D is a maximal orthogonal subset of A. Suppose that $D^{\perp\perp} \neq A$. Then there exists $z \in A$ such that $z \notin D^{\perp\perp}$. Evidently $z \notin D^{\perp}$, for otherwise $D \cup \{z\}$ would be an orthogonal subset of A. It follows from (ii) that there exists $w \in D^{\perp}$ such that $w \in (z^{\perp} \cap D^{\perp})^{\perp} \subset A$, contradicting the maximality of D. Hence, (ii) $\Rightarrow$ (iii).

Finally, suppose that (iii) holds. Let $A, B \in \mathscr{C}(Z)$ with $A \subset B$ and with $B \cap A^{\perp} = \varnothing$. To prove (i), it will suffice to show that $A = B$. Let D_1 be a maximal orthogonal subset of A. By Zorn's lemma, extend D_1 to a maximal orthogonal subset D_2 of B. By (iii), $D_1^{\perp\perp} = A$ and $D_2^{\perp\perp} = B$. If $D_1 = D_2$, we are finished. So, suppose that $d \in D_2 \setminus D_1$. Since D_2 is an orthogonal set, $d \in D_1{}^{\perp}$. Hence $d \in D_2^{\perp\perp} \cap D_1{}^{\perp} = A \cap B^{\perp}$, contradicting $A \cap B^{\perp} = \varnothing$, and the proof is complete.

3. The Main Theorem

In the present section, we let $(X, \#)$ denote any complete orthomodular space and we let Γ be the free monoid generated by X. Then, $(\Gamma, \perp)$ is an orthogonality space, where $\perp$ is the "lexicographic orthogonality" relation induced on Γ by $\#$ as in the introduction. Our task is to prove that $(\Gamma, \perp)$ is a complete orthomodular space.

If M and N are subsets of Γ, we write $MN = \{mn \mid m \in M \text{ and } n \in N\}$ by definition. We shall not bother to distinguish between the element $a \in \Gamma$ and the set $\{a\}$ in what follows, so that, for instance,

$$aM = \{am \mid m \in M\}.$$

If a, $b \in \Gamma$, then evidently $(ab)^{\perp} = ab^{\perp} \cup a^{\perp}$. Using this fact, one easily proves the following theorem:

THEOREM 2. *Let D be any orthogonal subset of Γ. Suppose that $D \neq \varnothing$ and that, for each $d \in D$, M_d is a non-empty subset of Γ. Let*

$$D_0 = \{d \in D \mid M_d{}^{\perp} \neq \varnothing\}.$$

Then:

(i) $[\bigcup\{dM_d \mid d \in D\}]^\perp = \bigcup\{dM_d^\perp \mid d \in D\} \cup D^\perp$.

(ii) $[\bigcup\{dM_d \mid d \in D\}]^{\perp\perp} = \bigcup\{dM_d^{\perp\perp} \mid d \in D_0\} \cup [D_0^\perp \cap D^{\perp\perp}]$.

We now define maps $\psi : \mathscr{C}(X, \#) \to \mathscr{C}(\Gamma, \perp)$ and

$$\psi^+ : \mathscr{C}(\Gamma, \perp) \to \mathscr{C}(X, \#)$$

as follows: For $M \in \mathscr{C}(X, \#)$, $\psi(M) = (M\Gamma)^{\perp\perp}$ and for $N \in \mathscr{C}(\Gamma, \perp)$, $\psi^+(N) = \{x \in X \mid x\Gamma \cap N \neq \varnothing\}^{\#\#}$. Direct calculation shows that for $M \in \mathscr{C}(X, \#)$ and for $N \in \mathscr{C}(\Gamma, \perp)$ we have the following:

(i) $\psi(M) = M\Gamma$ for $M \neq X$.

(ii) $\psi(X) = \Gamma$.

(iii) $\psi(M^\#) = M^\perp = (\psi(M))^\perp$.

(iv) $\psi(x^\#) = x^\perp$ and $\psi(x^{\#\#}) = x^{\perp\perp}$ for all $x \in X$.

(v) $\psi^+(\psi(M)) = M$.

(vi) $\psi(\psi^+(N)) \supset N$.

Since ψ and ψ^+ are both isotone maps, the theory of residuated maps together with (i)–(vi) above shows that $\psi : \mathscr{C}(X, \#) \to \mathscr{C}(\Gamma, \perp)$ is an injection which preserves arbitrary meets and joins and which preserves the orthocomplementation. (See [3].) Hence the complete ortho-lattice $\mathscr{C}(\Gamma, \perp)$ contains a subortho-lattice $\psi(\mathscr{C}(X, \#))$ that is orthoisomorphic to the complete orthomodular lattice $\mathscr{C}(X, \#)$.

LEMMA 3. *Let D be an orthogonal subset of Γ with $D, D^\perp \neq \varnothing$. Let $y \in X$ be such that $D^\perp \cap (y^\perp \cap D^\perp)^\perp = \varnothing$. Then $y \in D^{\perp\perp}$.*

Proof. Let $I = \{x \in X \mid x\Gamma \cap D \neq \varnothing\}$ and define $D_x = \{b \in \Gamma \mid xb \in D\}$ for each $x \in I$. Let $J = \{x \in I \mid D_x^\perp \neq \varnothing\} \cap y^\perp$ and put $K = I \setminus J$.

Since $D, D^\perp \neq \varnothing$, then I is a non-empty orthogonal subset of X and $D = \bigcup\{xD_x \mid x \in I\}$. By Theorem 2, $D^\perp = \bigcup\{xD_x^\perp \mid x \in I\} \cup I^\perp$, so that $y^\perp \cap D^\perp = \bigcup\{xD_x^\perp \mid x \in J\} \cup (y^\perp \cap I^\perp)$. Hence

$$(y^\perp \cap D^\perp)^\perp = \cup\{xD_x^{\perp\perp} \mid x \in J\} \cup [J^\perp \cap (y^\perp \cap I^\perp)^\perp].$$

Using the orthomonomorphism $\psi : \mathscr{C}(X, \#) \to \mathscr{C}(\Gamma, \perp)$ and the fact that $\mathscr{C}(X, \#)$ is an orthomodular lattice, we see that

$$\begin{aligned} J^\perp \cap (y^\perp \cap I^\perp)^\perp &= \psi[J^\# \wedge (y^{\#\#} \vee I^{\#\#})] \\ &= \psi(y^{\#\#} \vee K^{\#\#}) \\ &= (y^\perp \cap K^\perp)^\perp. \end{aligned}$$

Note that, if $x \in K$, then $x^{\perp\perp} \subset K^{\perp\perp} \subset (y^{\perp} \cap K^{\perp})^{\perp} \subset (y^{\perp} \cap D^{\perp})^{\perp}$; hence $xD_x^{\perp} \subset D^{\perp} \cap x^{\perp\perp} \subset D^{\perp} \cap (y^{\perp} \cap D^{\perp})^{\perp} = \varnothing$. It follows that $K = \{x \in I \mid D_x^{\perp} = \varnothing\}$.

Since $K = I \setminus J$ and since I is an orthogonal set, then $K \subset J^{\perp}$. The latter inclusion, together with the fact that $y \in J^{\perp}$, gives $(y^{\perp} \cap K^{\perp})^{\perp} \subset J^{\perp}$; hence

$$I^{\perp} \cap (y^{\perp} \cap K^{\perp})^{\perp} = K^{\perp} \cap J^{\perp} \cap (y^{\perp} \cap K^{\perp})^{\perp} = K^{\perp} \cap (y^{\perp} \cap K^{\perp})^{\perp}.$$

Since $I^{\perp} \subset D^{\perp}$ and $(y^{\perp} \cap K^{\perp})^{\perp} \subset (y^{\perp} \cap D^{\perp})^{\perp}$, then we have

$$K^{\perp} \cap (y^{\perp} \cap K^{\perp})^{\perp} = I^{\perp} \cap (y^{\perp} \cap K^{\perp})^{\perp} \subset D^{\perp} \cap (y^{\perp} \cap D^{\perp})^{\perp} = \varnothing.$$

Using the orthomonomorphism ψ again, and that $K^{\perp} \cap (y^{\perp} \cap K^{\perp})^{\perp} = \varnothing$, we see that $K^{\#} \wedge (y^{\#\#} \vee K^{\#\#}) = \varnothing$. Because of the orthomodularity of $\mathscr{C}(X, \#)$, the latter implies $y^{\#\#} \subset K^{\#\#}$. Application of the map ψ gives $y^{\perp\perp} \subset K^{\perp\perp}$, $y \in K^{\perp\perp}$. By part (ii) of Theorem 2,

$$K^{\perp\perp} = I^{\perp\perp} \cap \{x \in I \mid D_x^{\perp} \neq \varnothing\}^{\perp} \subset D^{\perp\perp};$$

hence $y \in D^{\perp\perp}$ as desired.

THEOREM 4. *If $(X, \#)$ is a complete orthomodular space, so is $(\Gamma, \perp)$.*

Proof. Suppose not. By part (ii) of Theorem 1, there exists a pair (D, a) consisting of an orthogonal subset D of Γ and an element $a \in \Gamma$ such that $a \notin D^{\perp} \cup D^{\perp\perp}$ and $\varnothing = D^{\perp} \cap (a^{\perp} \cap D^{\perp})^{\perp}$. Call such a pair improper. Evidently $D \neq \varnothing$ and $a^{\perp} \cap D^{\perp} \neq \varnothing$ for an improper pair (D, a). Consequently $a \neq 1$. Hence we have $a = x_1 x_2 \cdots x_n$ for suitable elements $x_1, x_2, \ldots, x_n \in X$. We define $n =$ length (a). Among all improper pairs, choose one, (D, a), for which length (a) is minimal. We can write $a = yb$, $y \in X$, $b \in \Gamma$. Evidently, there exists no improper pair (B, b).

Let $I = \{x \in X \mid x\Gamma \cap D \neq \varnothing\}$. Since $D \neq \varnothing$ and since $1 \notin D$, then I is a non-empty orthogonal subset of X and we have $D = \bigcup\{xD_x \mid x \in I\}$, where, for $x \in I$, $D_x = \{c \in \Gamma \mid xc \in D\}$ is a non-empty orthogonal subset of Γ. By Theorem 2, $D^{\perp} = \bigcup\{xD_x^{\perp} \mid x \in I\} \cup I^{\perp}$. Since $a^{\perp} = yb^{\perp} \cup y^{\perp}$, then $\varnothing \neq a^{\perp} \cap D^{\perp} = M_1 \cup M_2 \cup M_3$, where $M_1 = I^{\perp} \cap yb^{\perp}$, $M_2 = \bigcup\{xD_x^{\perp} \cap yb^{\perp} \mid x \in I\}$ and $M_3 = D^{\perp} \cap y^{\perp}$.

If $M_1 \neq \varnothing$, then $y \in I^{\perp}$, so $a = yb \in I^{\perp} \subset D^{\perp}$, a contradiction. Hence $M_1 = \varnothing$. If $M_2 \neq \varnothing$, then $y \in I$ and $D_y^{\perp} \cap b^{\perp} \neq \varnothing$. It follows that $yD_y^{\perp} \subset D^{\perp}$ and $y(b^{\perp} \cap D_y^{\perp})^{\perp} \subset (a^{\perp} \cap D^{\perp})^{\perp}$; hence that

$$y(D_y^{\perp} \cap (b^{\perp} \cap D_y^{\perp})^{\perp}) \subset D^{\perp} \cap (a^{\perp} \cap D^{\perp})^{\perp} = \varnothing,$$

so $D_y^{\perp} \cap (b^{\perp} \cap D_y^{\perp})^{\perp} = \varnothing$. Since the pair (D_y, b) cannot be improper, we must have $b \in D_y^{\perp} \cup D_y^{\perp\perp}$. If $b \in D_y^{\perp}$, then $a = yb \in D^{\perp}$, a contradiction. Hence $b \in D_y^{\perp\perp}$. Since $D_y^{\perp} \cap b^{\perp} \neq \varnothing$, then $D_y^{\perp} \neq \varnothing$, so $yD_y^{\perp\perp} \subset D^{\perp\perp}$ by part (ii) of Theorem 2. Thus we obtain the contradiction $a = yb \in D^{\perp\perp}$. Conclusion: $M_2 = \varnothing, a^{\perp} \cap D^{\perp} = y^{\perp} \cap D^{\perp}, D^{\perp} \cap (y^{\perp} \cap D^{\perp})^{\perp} = \varnothing$ and so $y \in D^{\perp\perp}$ by Lemma 3. Since $y \in D^{\perp\perp}$, then $a = yb \in D^{\perp\perp}$ contradicting our original assumption.

4. Concluding Remarks

The complete orthomodular lattices $\mathscr{C}(\Gamma, \perp)$ that arise from the complete orthomodular spaces $(X, \#)$ as in Section 3 are combinatorially quite rich. For instance, if one starts with a two-point orthogonality space $X = \{a, b\}$ with $a \# b$, then the orthomodular lattice $\mathscr{C}(\Gamma, \perp)$ turns out to be isomorphic to the Boolean algebra of all Borel subsets of the unit interval modulo the σ-ideal of all meager Borel sets. More generally, if $\mathscr{C}(X, \#)$ is a Boolean algebra, then $\mathscr{C}(\Gamma, \perp)$ will be a Boolean algebra.

On the other hand, one can show that, if the orthomodular lattice $\mathscr{C}(X, \#)$ is simple (that is, has no non-trivial orthohomomorphic images), then so is $\mathscr{C}(\Gamma, \perp)$; hence we have the means available for constructing a rather large class of simple complete orthomodular lattices.

In empirical logic [4], one is primarily interested in the special case in which the base space $(X, \#)$ is a disjoint sum of two or more non-trivial classic orthogonality spaces. Then, $\mathscr{C}(\Gamma, \perp)$ is called the *complete operational logic* over $(X, \#)$. These complete operational logics have many interesting features—for instance, they are simple, their only modular pairs are commuting pairs, and their automorphisms are induced by $\#$-preserving bijections of the base space.

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